Exercise 1: Invert the thermo-elastic constitutive relation of strains in terms of stresses,

$$\varepsilon_{ij} = \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\lambda}{3\lambda+2\mu}\sigma_{kk}\delta_{ij}\right) + \alpha\left(T-T_0\right)\delta_{ij} \qquad \text{(a)}$$

to obtain the thermos-elastic relation of stresses in terms of strains,

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij} - (3\lambda+2\mu)\alpha\left(T-T_0\right)\delta_{ij} \quad . \qquad \text{(b)}$$

Solution:

From (a) we have

$$\varepsilon_{ii} = \frac{1}{2\mu}\left(\sigma_{ii} - \frac{\lambda}{3\lambda+2\mu}\sigma_{kk}\delta_{ii}\right) + \alpha\left(T-T_0\right)\delta_{ii} = \frac{1}{2\mu}\left(\sigma_{ii} - \frac{3\lambda}{3\lambda+2\mu}\sigma_{kk}\right) + 3\alpha\left(T-T_0\right)$$

$$= \frac{1}{2\mu}\left(\frac{(3\lambda+2\mu)\sigma_{ii} - 3\lambda\sigma_{kk}}{3\lambda+2\mu}\right) + 3\alpha\left(T-T_0\right) = \frac{1}{2\mu}\left(\frac{(3\lambda+2\mu)\sigma_{kk} - 3\lambda\sigma_{kk}}{3\lambda+2\mu}\right) + 3\alpha\left(T-T_0\right)$$

$$= \frac{1}{2\mu}\frac{2\mu\sigma_{kk}}{3\lambda+2\mu} + 3\alpha\left(T-T_0\right) = \frac{\sigma_{kk}}{3\lambda+2\mu} + 3\alpha\left(T-T_0\right)$$

$$\Rightarrow \sigma_{kk} = (3\lambda+2\mu)\varepsilon_{ii} - 3\alpha(3\lambda+2\mu)\left(T-T_0\right) = (3\lambda+2\mu)\left[\varepsilon_{ii} - 3\alpha\left(T-T_0\right)\right]$$

Introduce the last expression into (a) and solve for the stress

$$\varepsilon_{ij} = \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\lambda}{3\lambda+2\mu}(3\lambda+2\mu)\left(\varepsilon_{ii} - 3\alpha\left(T-T_0\right)\right)\delta_{ij}\right) + \alpha\left(T-T_0\right)\delta_{ij}$$

$$\varepsilon_{ij} = \frac{1}{2\mu}\left[\sigma_{ij} - \lambda\varepsilon_{ii}\delta_{ij} + 3\lambda\alpha\left(T-T_0\right)\delta_{ij}\right] + \alpha\left(T-T_0\right)\delta_{ij}$$

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{ii}\delta_{ij} - (3\lambda+2\mu)\alpha\left(T-T_0\right)\delta_{ij}$$

The last expression is relation (b).

Exercise 2: Use the stress and strain relations in Exercise 1 to express the strain energy density

$$W = \frac{1}{2}\varepsilon_{ij}\sigma_{ij}$$

in terms of strains for a thermoelastic solid.

Solution: Introduce the stresses in the expression for energy and carry out the calculations,

$$\frac{1}{2}\varepsilon_{ij}\sigma_{ij} = \frac{1}{2}\varepsilon_{ij}\left[2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij} - (3\lambda+2\mu)\alpha\left(T-T_0\right)\delta_{ij}\right]$$

$$= \frac{2}{2}\mu\varepsilon_{ij}\varepsilon_{ij} + \frac{\lambda}{2}\varepsilon_{kk}\varepsilon_{ij}\delta_{ij} - \frac{(3\lambda+2\mu)}{2}\alpha\left(T-T_0\right)\varepsilon_{ij}\delta_{ij}$$

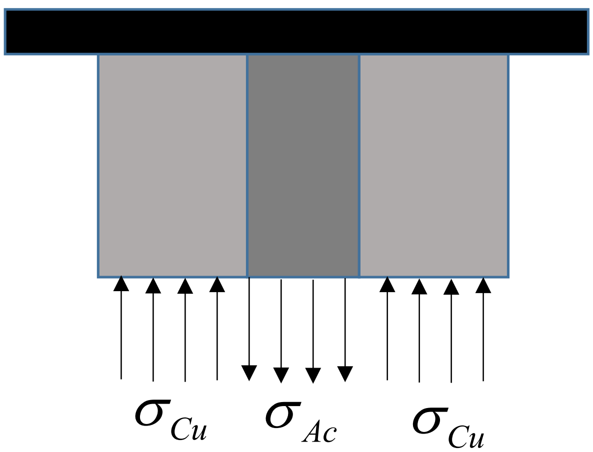
$$= \mu\varepsilon_{ij}\varepsilon_{ij} + \frac{\lambda}{2}\varepsilon_{kk}\varepsilon_{ii} - \frac{(3\lambda+2\mu)}{2}\alpha\left(T-T_0\right)\varepsilon_{ii}$$

$$\Rightarrow W(\varepsilon_{ij}) = \mu\varepsilon_{ij}\varepsilon_{ij} + \frac{\lambda}{2}\varepsilon_{kk}\varepsilon_{ii} - \frac{(3\lambda+2\mu)}{2}\alpha\left(T-T_0\right)\varepsilon_{ii}$$

Exercise 3: Calculate the stresses in the cupper (*Cu*) and steel (Ac) stress as well as the changes in the height Δh of the two rigid plates when the temperature of the assembly in the Figure is increased by ΔT.
During the thermal loading:
(a) there is no friction along the interface between the two materials. (b) two rigid plates remain parallel. (c) we consider $h >> d$ and the horizontal stresses negligible.

The sections and the material properties are known:
$Cu\,(\alpha_1, E_1, F_1)$, $Ac\,(\alpha_2, E_2, F_2)$ with $\alpha_1 > \alpha_2$.

Solution: Due to the differences in thermal expansion coefficients and the constraints of the two rigid parallel plates, stresses will develop in the assembly. Due to $h >> d$, the loading is considered uniaxial. The developed thermal stresses are taken uniform in a typical section away from the rigid plates as shown in the Figure below.



For simplicity we set

$$F_{Cu} \to F_1;\quad \alpha_{Cu} \to \alpha_1;\quad E_{Cu} \to E_1$$
$$F_{Ac} \to F_2;\quad \alpha_{Ac} \to \alpha_2;\quad E_{Ac} \to E_2$$
$$\sigma_{Cu} \to \sigma_1$$
$$\sigma_{Ac} \to \sigma_2$$

The following two equations are relevant:

Compatibility of Deformation: The vertical change in length for each material is the same due to the parallel constraints:

$$\Delta h = \left(\frac{\sigma_2}{E_2} h + \alpha_2 h \Delta T\right)_{Ac} = \left(-\frac{\sigma_1}{E_1} h + \alpha_1 h \Delta T\right)_{Cu} \qquad (1)$$

Force equilibrium:

$$\sigma_2 F_2 - \sigma_1 F_1 = 0 \qquad (2)$$

From (1)

$$\frac{\sigma_1}{E_1} + \frac{\sigma_2}{E_2} = (\alpha_1 - \alpha_2)\Delta T \qquad (3)$$

From (2)

$$\sigma_1 = \sigma_2 \frac{F_2}{F_1} \qquad (4)$$

Introduce (4) in (3)

$$\frac{\sigma_2}{E_1}\frac{F_2}{F_1} + \frac{\sigma_2}{E_2} = (\alpha_1 - \alpha_2)\Delta T \Rightarrow \frac{\sigma_2}{E_2}\left(1 + \frac{E_2}{E_1}\frac{F_2}{F_1}\right) = \frac{\sigma_2}{E_2}(1+m) = (\alpha_1 - \alpha_2)\Delta T\,,$$

where $m = \frac{E_2}{E_1}\frac{F_2}{F_1}$

$$\sigma_2 = (\alpha_1 - \alpha_2)\frac{E_2}{1+m}\Delta T$$ Stress proportional to the temperature changes.

From (2)

$$\sigma_1 = (\alpha_1 - \alpha_2)\frac{E_2}{1+m}\frac{F_2}{F_1}\Delta T = (\alpha_1 - \alpha_2)\frac{E_1}{1+m}\frac{F_2 E_2}{F_1 E_1}\Delta T$$

$$\Rightarrow \sigma_1 = (\alpha_1 - \alpha_2)\frac{mE_1}{1+m}\Delta T$$ Stress proportional to the temperature changes.

From (1) for *Ac*:

$$\Delta h = \left(\frac{\sigma_2}{E_2} + \alpha_2 \Delta T\right)h = \left((\alpha_1 - \alpha_2)\frac{E_2}{1+m}\frac{1}{E_2}\Delta T + \alpha_2 \Delta T\right)h\,.$$

$$\Delta h = \left(\alpha_2 + (\alpha_1 - \alpha_2)\frac{1}{1+m}\right)\Delta T h$$

Similarly for *Cu*:

$$\Delta h = \left(\alpha_1 - (\alpha_1 - \alpha_2)\frac{m}{1+m}\right)\Delta T h\,.$$